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Finite-size Analysis of $O(N)$ Nonlinear σ Model on Semi-compact Spaces

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Abstract

Fisher's phenomenological renormalization method is used to calculate the mass gap and the correlation length of the $O(N)$ nonlinear σ model on a semi-compact space $S^1 \times \mathbf{R}^2$. This shows that the ultraviolet momentum cut-off does not conflict with the infrared cut-off along the S^1 direction. The mass gap on $S^2 \times \mathbf{R}^1$ is also discussed.

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The $O(N)$ nonlinear σ (NLσ) model on a semi-compact space $S^1 \times \mathbf{R}^2$ has been intensively studied recently in the context of two-dimensional quantum spin systems[1] and in researching the critical phenomena in three dimensions[2][3]. The calculation of the partition functions and correlation length (or mass gap) is done with the ultraviolet (UV) momentum cut-off in the $\mathbf{R}^2$ direction and usually without any UV momentum cut-off along the S^1 direction being imposed[1][4]. This procedure is quite natural in the view of the imaginary time path integral method, which discritizes only the spatial coordinates and leaves the temporal one continuous. However, this scheme might conflict with the infrared (IR) cut-off along the S^1 direction because of its unrenormalizability.

In this paper, we consider the gap equation on the lattice and make use of Fisher's phenomenological renormalization (PR) method[5][6] to clarify the above point. The Pr method is a renormalization method in the real space based on a hypothesis of the pseudo-scale invariance for large but finite systems. In this scheme, we have the advantage that the UV cut-off does not appear explicitly. In terms of the PR method, the correlation length can be calculated exactly on $S^1 \times S^1 \times \mathbf{R}^1$ or $S^1 \times S^1 \times S^1$[5]. We apply the PR method to the $O(N)$ NLσ model on $S^1 \times \mathbf{R}^2$ and $S^2 \times \mathbf{R}^1$. These PR procedures reproduce the mass-gap obtained previously in the cut-off regularization[1]. Therefore, we consider that the UV cut-off regularization does not conflict with the IR cut-off along the S^1 or S^2 direction. Of course, it is widely believed that the physical quantities can be calculated independently of the regularization in renormalizable systems. Our results are consistent with this intuition despite its unrenormalizablilty in the ordinary meaning.

We consider the $O(N)$ NLσ model on a semi-compact space $S^1 \times \mathbf{R}^2$ with the radius of S^1, L. The partition function is given by

$$Z = \int D\vec{n}(x) \exp\left(-\frac{N}{2g}\int_{S^1\times\mathbf{R}^2} d^3x (\partial_\mu \vec{n}(x))^2\right), \tag{1}$$

where $\vec{n}(x)$ is an N-dimensional vector field normalized to $(\vec{n}(x))^2 = 1$. This model is believed to describe the long range behavior of the two-dimensional antiferromagnetic

Heisenberg model as

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j, \tag{2}$$

with large spin S at finite temperature. We now follow the PR method. To begin with we solve the constraint $(\vec{n}(x))^2 = 1$ by means of the auxiliary field as

$$\begin{aligned} Z &= \int D\vec{n}(x) D\mu(x) \exp\left(-\frac{N}{2g} \int_{S^1 \times \mathbf{R}^2} d^3x [(\partial_\mu \vec{n}(x))^2 + \mu(x)((\vec{n}(x))^2 - 1)] \right), \\ &= \int D\mu(x) \exp(-(N/2) S_{\mathit{eff}}), \end{aligned} \tag{3}$$

where we denote

$$S_{\mathit{eff}} = -\frac{1}{g} \int_{S^1 \times \mathbf{R}^2} d^3x \mu(x) + \log\det(-\partial^2 + \mu(x)). \tag{4}$$

We consider the large N limit, which enables us to make use of the saddle point method[7]. If we impose the periodic boundary condition in the direction of S^1, the gap equation on $S^1 \times \mathbf{R}^2$ with lattice regularization is given by

$$\begin{aligned} \frac{1}{g} &= \int \frac{d^3q}{(2\pi)^3} \frac{1}{\xi_L^{-2} + 2\sum_{\mu=0}^{2}(1 - \cos q_\mu)} \sum_{n \in \mathbf{Z}} (2\pi) \delta\left(q_0 - \frac{2\pi}{L} n \right) \\ &= \int \frac{d^3q}{(2\pi)^3} \frac{1}{\xi_L^{-2} + 2\sum_{\mu=0}^{2}(1 - \cos q_\mu)} \sum_{n \in \mathbf{Z}} e^{iq_0 nL}, \end{aligned} \tag{5}$$

where we apply Poisson's summation formula and $\xi_L = (\langle \mu(x) \rangle)^{-1/2}$ is denoted as the correlation length in $S^1 \times \mathbf{R}^2$, which is a function of L and g. We put $\xi_\infty = \lim_{L \to \infty} \xi_L$, which is divergent when the coupling constant g corresponds to the critical value g_c. Here we denote g_c as the UV fixed point of the renormalization group. If we subtract the equation (5) for ξ_∞ from that for ξ_L, the difference gives

$$\begin{aligned} &(\xi_\infty^{-2} - \xi_L^{-2}) \int \frac{d^3q}{(2\pi)^3} \frac{1}{\xi_\infty^{-2} + 2\sum_{\mu=0}^{2}(1 - \cos q_\mu)} \frac{1}{\xi_L^{-2} + 2\sum_{\mu=0}^{2}(1 - \cos q_\mu)} \\ &\qquad + \int \frac{d^3q}{(2\pi)^3} \sum_{n \neq 0} \frac{e^{iq_0 nL}}{\xi_L^{-2} + 2\sum_{\mu=0}^{2}(1 - \cos q_\mu)} = 0. \end{aligned} \tag{6}$$

If we choose the coupling constant g to be close to the critical value g_c, we can replace the propagator $1/(\xi_L^{-2} + \sum(1 - \cos q_\mu))$ by $1/(\xi_L^{-2} + q^2)$ and expand the momentum region

$[-\pi,\pi]$ to $(-\infty,\infty)$. Therefore the first term can be further simplified as

$$(\xi_\infty^{-2}-\xi_L^{-2})\int\frac{d^3q}{(2\pi)^3}\frac{1}{\xi_\infty^{-2}+q^2}\frac{1}{\xi_L^{-2}+q^2}=(\xi_\infty^{-2}-\xi_L^{-2})\frac{\Gamma(1/2)}{(4\pi)^{3/2}}\int_0^1\frac{dt}{(t\xi_L^{-2}+(1-t)\xi_\infty^{-2})^{1/2}}. \quad (7)$$

The second term can be also simplified as

$$\sum_{n\neq 0}\int\frac{d^3q}{(2\pi)^3}\frac{e^{iq_0nL}}{\xi_L^{-2}+q^2}=\frac{1}{L}\int_0^\infty dte^{-t(L/\xi_L)^2}u(t), \quad (8)$$

where we put $u(t)=\sum_{n\neq 0}e^{-n^2/4t}/(4\pi t)^{3/2}$. Therefore, the gap equation is reduced to

$$\left[\left(\frac{L}{\xi_\infty}\right)^2-\left(\frac{L}{\xi_L}\right)^2\right]\left(\frac{\xi_L}{L}\right)\frac{\Gamma(1/2)}{(4\pi)^{3/2}}\int_0^1 dt\frac{1}{\sqrt{(t(\xi_L/\xi_\infty)+(1-t))}}$$
$$+\int_0^\infty dte^{-t(L/\xi_L)^2}u(t)=0. \quad (9)$$

If we choose the coupling constant g to be the critical value g_c, the correlation length in $\mathbf{R}^3$ becomes infinite and the gap equation simplifies to

$$\sum_{n=1}^{\infty}\frac{(\xi_L/L)}{n}e^{-n/(\xi_L/L)}=-\frac{\xi_L}{L}\log(1-e^{-L/\xi_L})=\frac{1}{2}, \quad (10)$$

which reproduces the mass gap obtained in Ref.[1] as

$$\xi_L/L=1/(2\log\frac{\sqrt{5}+1}{2}). \quad (11)$$

From this mass gap, the finite size correction can be calculated as

$$F_L-F_\infty=\frac{4N}{5}\frac{\zeta(3)}{L^3}, \quad (12)$$

making use of the addition formula of Roger's polylogarithmic function[4].

We can also calculate the correlation length in this PR procedure as follows.

The UV stable critical coupling constant g_c in $\mathbf{R}^3$ is given by

$$\frac{1}{g_c}=\int_{\mathbf{R}^3}\frac{d^3q}{(2\pi)^3}\frac{1}{q^2}. \quad (13)$$

If we subtract Eq. (13) from Eq. (5), we obtain the equation

$$
\begin{aligned}
\frac{1}{g}-\frac{1}{g_c} &= -\frac{1}{\xi_L^2}\int\frac{d^3q}{(2\pi)^3}\frac{1}{q^2}\frac{1}{\xi_L^{-2}+q^2}+2\sum_{n=1}^{\infty}\int\frac{d^3q}{(2\pi)^3}\frac{e^{iq_0nL}}{\xi_L^{-2}+q^2}, \\
&= -\frac{1}{4\pi\xi_L}-\frac{1}{2\pi L}\log(1-e^{-L/\xi_L}), \qquad (14)
\end{aligned}
$$

which determines the correlation length ξ_L on $S^1\times\mathbf{R}^2$. We solve Eq.(14) in two regions of the coupling constant g in the large L (*i.e.* low temperature) limit. First, we consider the region $g \ll g_c$, which is called the *renormalized classical region* in Ref.[1]. In this region, the correlation length is given by

$$
\xi_L = L\exp\left(2\pi L\left(\frac{1}{g}-\frac{1}{g_c}\right)\right). \qquad (15)
$$

Second we consider the region $g \gg g_c$, which is called the *quantum disordered region* in Ref.[1]. In this region, the solution of Eq. (14) is

$$
\xi_L^{-1}=\frac{2\pi}{g_c}\left(1-\frac{g_c}{g}\right), \qquad (16)
$$

which is independent of the size of S^1, L. These expressions for the correlation lengths in the two regions correspond to those obtained by the momentum cut-off method[1].

From the above calculations, we can see that the UV momentum cut-off procedure does not conflict with the IR cut-off L in the case of the $O(N)$ NLσ model on $S^1\times\mathbf{R}^2$.

Furthermore, we can discuss the dependence of the correlation length in $S^2\times\mathbf{R}$, with R, which is the radius of S^2. In this case, denoting the correlation length with the radius R as ξ_R, the gap equation is given by

$$
\begin{aligned}
\frac{1}{g} &= \frac{1}{4\pi R^2}\sum_{n=0}^{\infty}\frac{2n+1}{\xi_R^{-2}+p^2+n(n+1)/R^2} \\
&= \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{(\xi_R^{-2}-\frac{1}{4R^2})+\vec{p}^2}+2\sum_{n=1}^{\infty}(-)^n\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{(\xi_R^{-2}-\frac{1}{4R^2})+\vec{p}^2}\cos(2\pi Rn|\mathbf{p}_\perp|). \qquad (17)
\end{aligned}
$$

Here we make a use of Poisson's summation formula again and denote the three-dimensional vector $\vec{p}=(\mathbf{p}_\perp,p)$. In a similar manner to the case of $S^1\times\mathbf{R}^2$, we subtract Eq. (17) for

$R=\infty$ from that for finite R. If we put the coupling constant g at the critical value g_c, the difference gives

$$-\frac{m}{4\pi}+2\sum_{n=1}^{\infty}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\vec{p}^2+m^2}(-)^n\cos(2\pi Rn|\mathbf{p}_\perp|)=0, \tag{18}$$

where we put $m^2=\xi_R^{-2}-(1/4R^2)$. The solution of Eq.(18) is [2]

$$m=0 \qquad i.e. \quad \xi_R=2R, \tag{19}$$

which is consistent with the expected value of the conformal coupling in the three-dimensional space[8]. The conformally invariant free scalar theory in a three-dimensional space M is given by

$$S=\frac{1}{2}\int_M[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\eta R^{(2)}\phi^2]\sqrt{g}d^3x, \tag{20}$$

where $\eta=1/8$ and $R^{(2)}$ is the scalar curvature of M. If $M=S^2\times\mathbf{R}$, the conformal coupling takes the value of $\eta R^{(2)}=1/2R$, which corresponds to the obtained value of ξ_R^{-1} in Eq. (19). Therefore with the coupling constant g_c, the $O(N)$ NLσ model is conformally invariant on $S^2\times\mathbf{R}$.

Even though we do not have the exact form of the correlation length on $S^2\times S^1$, even with the critical coupling constant g_c, we expect that the UV cut-off regularization method does not conflict with the IR cut-off in this case either, and therefore that the UV cut-off regularization is available for $S^2\times S^1$.

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